Sliding Book

1 Introduction

The goal is to get the robot to (1) throw a book onto a desk such that (2) the book slides on the desk, (3) to hit another book, (4) which falls down in free flight to fall into a basket on the ground. To perform this task, we will use the free flight equations written in freeFlight.pdf and some new equations for sliding over surfaces.

2 Motion Modeling

Previously, in freeFlight.pdf, we have defined the analytical trajectory function $F_F(t, t_0, q_0(u))$ which returns the state of a flying object starting with state $q_0(u)$ after $t - t_0$ time into the motion. u is the control variables that affect the initial position and velocity of the object.

In this study, we have multiple objects that will interact with each other where first one object performs free-flight and slides, then the two objects collide, and the second objects falls down. We are interested in the final state of the second object:

$$\begin{aligned} S(u) &= F_F(t_f(u), t_{c_2}(u), {}^2q^+_{t_{c_2}(u)}) \\ &= F_F(t_f(u), t_{c_2}(u), C_M({}^1q^-_{t_{c_2}(u)})) \\ &= F_F(t_f(u), t_{c_2}(u), C_M(F_S(t_{c_2}(u), t_{c_1}(u), C_S^D({}^1q^+_{t_{c_1}(u)})))) \\ &= F_F(t_f(u), t_{c_2}(u), C_M(F_S(t_{c_2}(u), t_{c_1}(u), C_S^D(F_F(t_{c_1}(u), t_0(u), {}^1q_0(u)))) \\ &= F_F \circ C_M \circ F_S \circ C_S^D \circ F_F \end{aligned} \tag{1}$$

where the notation represents:

- ${}^1q$ and ${}^2q$ refer respectively to the states of first and second objects
- t_{c_1} and t_{c_2} refer to the collisions times between first object and ground, and the two objects.
- $F_F(.)$ and $F_S(.)$ represent the free-flight and the sliding motions respectively
- ${}^2q^+ = C_M({}^1q^-)$ is the collision function for multiple objects - assumes elastic
- $q^+ = C_S^D(q^-)$ is the collision function for a single object with the ground where the vertical velocity component is dampened out.

and the following assumptions are made:

- u is the initial state of the first object and the final time.
- The initial position of the second object, ${}^2x^- = \left[\begin{smallmatrix}1&0&0&0\\0&1&0&0\end{smallmatrix}\right] {}^2q^-_{t_{c_2}(u)}$, prior to the collision, is known.
- $t_f(u) > t_{c_2}(u) > t_{c_1}(u) > t_0 = 0$.
- The objects are spheres with radius r. The table height is z_t.

Note that the goal is to compute the Jacobian $\frac{\partial S(u)}{\partial u}$ so that given some initial trajectory that has some collisions, the trajectory can be driven to fulfill a desired state $S(u)$ at time $t_f(u)$. Finding such a trajectory is a different concern at the moment.

2.1 Preliminaries

2.1.1 Sliding model F_S

Let the 3D state of an q_t be composed of a position $x_t \in \mathbb{R}^3$ and velocity $v_t \in \mathbb{R}^3$: $q_t = [x_t; q_t]^T$. We assume no rotations. From basic physics, we know how to update the position and velocity of an object which moves under constant acceleration a:

$$x_t = x_0 + tv_0 + \tfrac{1}{2}at^2 \tag{2}$$

$$v_t = v_0 + at \tag{3}$$

Now, for a sliding motion under friction acceleration is defined is computed as:

$$a = F_{net}/m = (0 - F_{friction}) \tag{4}$$

$$= (-N\mu)/m = ((-mg)\mu)/m = -g\mu \tag{5}$$

Using Equations 1 and 2, and the definition of acceleration, we can define the sliding function $F_S(.)$:

$$F_S(t, t_0, q_0) = q_0 + (t_f - t_0)\begin{bmatrix} 000100 \\ 000010 \\ 000001 \\ 000000 \\ 000000 \\ 000000 \end{bmatrix} q_0 + \begin{bmatrix} -\frac{1}{2}g\mu(t_f-t_0)^2 \\ -\frac{1}{2}g\mu(t_f-t_0)^2 \\ 0 \\ -g\mu(t_f-t_0) \\ -g\mu(t_f-t_0) \\ 0 \end{bmatrix} \tag{6}$$

where we will denote the velocity matrix as $\mathbb{M}$.

2.1.2 Collision of multiple objects: C_M

Let ${}^1v^-$ and ${}^2v^-$ be the velocities of the two objects before the collision, and ${}^1v^+$ and ${}^2v^+$ after the collision. Let ${}^1x^-$ and ${}^2x^-$ be the positions of the two objects. Only the component of the velocity of the first object that passes through the sphere centers is accounted for. Let $d = ({}^2x^- - {}^1x^-)/2r$ be the vector that goes from the first object center to the second. Following the conversation of momentum for an elastic collision, the velocity of the second object after the collision can be written as:

$${}^2v^+ = \frac{2\,{}^1m(d \cdot {}^1v^-)d}{{}^1m + {}^2m} \tag{7}$$

$$= \frac{2\,{}^1m\left[\frac{({}^2x^- - {}^1x^-)}{2r} \cdot {}^1v^-\right]\frac{({}^2x^- - {}^1x^-)}{2r}}{{}^1m + {}^2m} \tag{8}$$

$$= \frac{{}^1m}{2r^2({}^1m + {}^2m)}\left[({}^2x^- - {}^1x^-) \cdot {}^1v^-\right]({}^2x^- - {}^1x^-) \tag{9}$$

where ${}^1m$ and ${}^2m$ represent the masses of the objects. Remembering ${}^1x$ and ${}^1v$ are outputs of the sliding motion and ${}^2x^-$ is predefined, we can define C_M as:

$$C_M({}^1q^-) = \begin{bmatrix} 100 \\ 010 \\ 001 \\ 000 \\ 000 \\ 000 \end{bmatrix} {}^2x^- + \frac{{}^1m}{2r^2({}^1m+{}^2m)} \begin{bmatrix} 000 \\ 000 \\ 000 \\ 100 \\ 010 \\ 001 \end{bmatrix} \left[({}^2x^- - (\begin{bmatrix} 100000 \\ 010000 \\ 001000 \end{bmatrix} {}^1q^-)) \cdot (\begin{bmatrix} 000100 \\ 000010 \\ 000001 \end{bmatrix} {}^1q^-)\right]({}^2x^- - (\begin{bmatrix} 100000 \\ 010000 \\ 001000 \end{bmatrix} {}^1q^-)) \tag{10}$$

where $k_{C_M} = \frac{{}^1m}{2r^2({}^1m+{}^2m)}$ will later be used for brevity.

2.1.3 Collision time of multiple objects: $t_{c_2}(u)$

Let ${}^2x^- = [{}^2x_0, {}^2y_0]^\intercal$. Assuming that after object 1 starts sliding, it would reach the second object, we can write the following equality:

$$||({}^2x^-) - \left[\begin{smallmatrix}1000\\0100\end{smallmatrix}\right] F_S(t_{c_2}(u), t_{c_1}(u), C_S^D(.))|| = 2r \tag{11}$$

Let $p^x_{C_S^D}$ and $v^x_{C_S^D}$ be the state variables of state after the first dampened collision $C_S^D(.)$. We rewrite:

$$\left[{}^2x_0 - \left(p^x_{C_S^D} + (t_{c_2} - t_{c_1})v^x_{C_S^D} + \tfrac{1}{2}g\mu(t_{c_2} - t_{c_1})^2\right)\right]^2 + \left[{}^2y_0 - \left(p^y_{C_S^D} + (t_{c_2} - t_{c_1})v^y_{C_S^D} + \tfrac{1}{2}g\mu(t_{c_2} - t_{c_1})^2\right)\right]^2 = 4r^2 \tag{12}$$

2.1.4 Dampened collision with ground: C_S^D

The idea is that we don't want the ball to bounce and just maintain its horizontal velocity. So, if v^- is the velocity of the object before the collision, we want its z-component to be zero:

$$C_S^D(q^-) = \begin{bmatrix}100000\\010000\\001000\\000100\\000010\\000000\end{bmatrix} q^- \tag{13}$$

where we will refer to the matrix as $\mathbb{M}_C$.

2.1.5 Collision time with table: $t_{c_1}(u)$

For the first ball to hit the table, it needs to reach the height $(z_t - r)$ to make contact with the top of the table at height z_t. Using free flight model $F_F(.)$, we can compute the flight time $t_{c-1}(u) - t_0 = t_{c-1}(u) - 0 = t_{c-1}(u)$. Note that p_0^z and v_0^z are the initial height and vertical velocity of the ball, both elements of the input u. The collision time $t_{c_1}(u)$ then is the solution to the following quadratic function:

$$p_0^z + v_0^z t_{c_1}(u) - 0.5 g t_{c_1}^2(u) = 0 \tag{14}$$

2.2 Computation of the Gradient

2.2.1 Chain Rule

We are interested in computing $\frac{\partial S(u)}{\partial u}$. Note that $\hat{e}_i$ is a 1×7 vector with zero elements but i^{th} being 1.

$$\frac{\partial S(u)}{\partial u} = \frac{\partial F_F(t_f(u), t_{c_2}(u), C_M(.))}{\partial u} = \frac{\partial\left[C_M(.) + \mathbb{M}C_M(.)(t_f(u) - t_{c_2}(u)) + \begin{bmatrix}0\\0\\-\frac{1}{2}g(t_f(u)-t_{c_2}(u))^2\\0\\0\\-g(t_f(u)-t_{c_2}(u))\end{bmatrix}\right]}{\partial u} \tag{15}$$

$$= \frac{\partial C_M(.)}{\partial u} + \frac{\partial\left[\mathbb{M}C_M(.)(t_f(u)-t_{c_2}(u))\right]}{\partial u} + \frac{\partial\begin{bmatrix}0\\0\\-\frac{1}{2}g(t_f(u)-t_{c_2}(u))^2\\0\\0\\-g(t_f(u)-t_{c_2}(u))\end{bmatrix}}{\partial u} \tag{16}$$

$$= \frac{\partial C_M(.)}{\partial u} + \left[\frac{\partial(\mathbb{M}C_M(.))}{\partial u}(t_f(u) - t_{c_2}(u)) + (\mathbb{M}C_M(.))\frac{\partial(t_f(u)-t_{c_2}(u))}{\partial u}\right] + \begin{bmatrix}0_{1\times 7}\\0_{1\times 7}\\\partial[-\frac{1}{2}g(t_f(u) - t_{c_2}(u))^2]/\partial u\\0_{1\times 7}\\0_{1\times 7}\\\partial[-g(t_f(u) - t_{c_2}(u))]/\partial u\end{bmatrix} \tag{17}$$

$$= \frac{\partial C_M(.)}{\partial u} + \mathbb{M}\left[\frac{\partial C_M(.)}{\partial u}(t_f(u) - t_{c_2}(u)) + C_M(.)\left(\hat{e}_7^\intercal - \frac{\partial t_{c_2}(u)}{\partial u}\right)\right] - \begin{bmatrix}0_{1\times 7}\\0_{1\times 7}\\g(t_f(u)-t_{c_2}(u))\partial(t_f(u) - t_{c_2}(u))/\partial u\\0_{1\times 7}\\0_{1\times 7}\\g\partial(t_f(u) - t_{c_2}(u))/\partial u\end{bmatrix} \tag{18}$$

$$= \frac{\partial C_M(.)}{\partial u} + \mathbb{M}\left[\frac{\partial C_M(.)}{\partial u}(t_f(u) - t_{c_2}(u)) + C_M(.)\left(\hat{e}_7^\intercal - \frac{\partial t_{c_2}(u)}{\partial u}\right)\right] - \begin{bmatrix}0_{1\times 7}\\0_{1\times 7}\\g(t_f(u)-t_{c_2}(u))\left(\hat{e}_7^\intercal - \partial t_{c_2}(u)/\partial u\right)\\0_{1\times 7}\\0_{1\times 7}\\g\left(\hat{e}_7^\intercal - \partial t_{c_2}(u)/\partial u\right)\end{bmatrix} \tag{19}$$

2.2.2 Computing $\frac{\partial t_{c_2}(u)}{\partial u}$:

We take the partial derivative of the distance function in Equation 12:

$$0 = \frac{\partial \left[\left\| {}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(C)) \right\|^2 - 4r^2 \right]}{\partial u} \tag{20}$$

$$= 2\left[{}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(C)) \right]^\intercal \frac{\partial \left[{}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(C) \right]}{\partial u} \tag{21}$$

$$= -2\left[{}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(C)) \right]^\intercal \mathbb{M}_{\mathbb{P}} \frac{\partial F_S(C)}{\partial u} \tag{22}$$

To preserve sanity, we first compute the partial derivative of the sliding function. Also let $\hat{e}_{xy} = \hat{e}_x + \hat{e}_y$.

$$\frac{\partial F_S(u)}{\partial u} = \frac{\partial \left[C_S^D(.) + \mathbb{M} C_S^D(.)(t_{c_2} - t_{c_1}) - \frac{1}{2} g\mu (t_{c_2} - t_{c_1})^2 \hat{e}_{12} - g\mu (t_{c_2} - t_{c_1}) \hat{e}_{45} \right]}{\partial u} \tag{23}$$

$$= \frac{\partial [\mathbb{M}_{\mathbb{P}} C_S^D(.)]}{\partial u} + \frac{\partial [\mathbb{M} C_S^D(.)(t_{c_2} - t_{c_1})]}{\partial u} - \frac{\partial \left[\frac{1}{2} g\mu (t_{c_2} - t_{c_1})^2 \hat{e}_{12} \right]}{\partial u} - \frac{\partial \left[g\mu (t_{c_2} - t_{c_1}) \hat{e}_{45} \right]}{\partial u} \Bigg] \tag{24}$$

$$= \frac{\partial C_S^D(.)}{\partial u} + \left(\frac{\partial [\mathbb{M} C_S^D(.)]}{\partial u} (t_{c_2} - t_{c_1}) + (\mathbb{M} C_S^D(.)) \frac{\partial (t_{c_2} - t_{c_1})}{\partial u} \right) - g\mu (t_{c_2} - t_{c_1}) \hat{e}_{12} \frac{\partial (t_{c_2} - t_{c_1})}{\partial u} - g\mu \hat{e}_{45} \frac{\partial (t_{c_2} - t_{c_1})}{\partial u} \tag{25}$$

$$= \frac{\partial C_S^D(.)}{\partial u} + \mathbb{M} \frac{\partial C_S^D(.)}{\partial u} (t_{c_2} - t_{c_1}) + \left(\mathbb{M} C_S^D(.) - g\mu (t_{c_2} - t_{c_1}) \hat{e}_{12} - g\mu \hat{e}_{45} \right) \frac{\partial (t_{c_2} - t_{c_1})}{\partial u} \tag{26}$$

$$= \left(\mathbf{I}_{6\times 6} + \mathbb{M}(t_{c_2} - t_{c_1}) \right) \frac{\partial C_S^D(.)}{\partial u} + \left(\mathbb{M} C_S^D(.) - g\mu ((t_{c_2} - t_{c_1}) \hat{e}_{12} + \hat{e}_{45}) \right) \frac{\partial (t_{c_2} - t_{c_1})}{\partial u} \tag{27}$$

In its simplest form, we can back substitute $\frac{\partial F_S(u)}{\partial u}$ to Equation 22 to obtain:

$$0 = -2\left[{}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(C)) \right]^\intercal \mathbb{M}_{\mathbb{P}} \frac{\partial F_S(C)}{\partial u} \tag{28}$$

$$= -2\left[{}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(C)) \right]^\intercal \mathbb{M}_{\mathbb{P}} \left[\left(\mathbf{I}_{6\times 6} + \mathbb{M}(t_{c_2} - t_{c_1}) \right) \frac{\partial C_S^D(.)}{\partial u} + \left(\mathbb{M} C_S^D(.) - g\mu ((t_{c_2} - t_{c_1}) \hat{e}_{12} - \hat{e}_{45}) \right) \frac{\partial (t_{c_2} - t_{c_1})}{\partial u} \right] \tag{29}$$

$$= -2c_1 \left[c_2 \frac{\partial C_S^D(.)}{\partial u} + c_3 \frac{\partial t_{c_2}}{\partial u} - c_3 \frac{\partial t_{c_1}}{\partial u} \right] \tag{30}$$

where in the last step, we replace the coefficients with smaller terms c_1 (1×6), c_2 (6×7) and c_3 (6×1) for brevity sake. Now, we can compute the partial derivative:

$$-2c_1 \left[c_2 \frac{\partial C_S^D(.)}{\partial u} - c_3 \frac{\partial t_{c_1}}{\partial u} \right] - 2c_1 c_3 \frac{\partial t_{c_2}}{\partial u} = 0 \tag{31}$$

$$c_1 c_3 \frac{\partial t_{c_2}}{\partial u} = -c_1 \left[c_2 \frac{\partial C_S^D(.)}{\partial u} - c_3 \frac{\partial t_{c_1}}{\partial u} \right] \tag{32}$$

$$\frac{\partial t_{c_2}(u)}{\partial u} = -\frac{c_1 \left[c_2 \frac{\partial C_S^D(.)}{\partial u} - c_3 \frac{\partial t_{c_1}(u)}{\partial u} \right]}{c_1 c_3} \tag{33}$$

2.2.3 Computing $\frac{\partial t_{c_1}(u)}{\partial u}$:

We take the partial derivative of Equation 12 that concerns the table height and the free-flight of the first object:

$$\frac{\partial p_0^z}{\partial u} + \frac{[v_0^z t_{c_1}(u)]}{\partial u} - \frac{\partial[1/2 g t_{c_1}^2(u)]}{\partial u} = 0 \tag{34}$$

$$\hat{e}_3 + \left(\frac{\partial v_0^z}{\partial u} t_{c_1}(u) + v_0^z \frac{\partial t_{c_1}(u)}{\partial u}\right) - g t_{c_1}(u) \frac{\partial t_{c_1}(u)}{\partial u} = 0 \tag{35}$$

$$\hat{e}_3 + \hat{e}_6 t_{c_1}(u) + \left(v_0^z - g t_{c_1}(u)\right) \frac{\partial t_{c_1}(u)}{\partial u} = 0 \tag{36}$$

$$\frac{\partial t_{c_1}(u)}{\partial u} = -\frac{\hat{e}_3 + \hat{e}_6 t_{c_1}(u)}{v_0^z - g t_{c_1}(u)} \tag{37}$$

2.2.4 Computing $\frac{\partial C_M(F_S(.))}{\partial u}$:

We take the partial derivation of Equation 10:

$$\frac{\partial C_M(F_S(.))}{\partial u} = \frac{\partial\left[\mathbb{M}_{\mathbb{P}}^{\top} {}^2x^- + k_{C_M} \mathbb{M}_{\mathbb{V}}^{\top}({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.))[({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.)) \cdot (\mathbb{M}_{\mathbb{V}} F_S(.))]\right]}{\partial u} \tag{38}$$

$$= k_{C_M} \mathbb{M}_{\mathbb{V}}^{\top} \frac{\partial\left[({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.))[({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.)) \cdot (\mathbb{M}_{\mathbb{V}} F_S(.))]\right]}{\partial u} \tag{39}$$

$$= k_{C_M} \mathbb{M}_{\mathbb{V}}^{\top}\left[\frac{\partial[{}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.)]}{\partial u}[({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.)) \cdot (\mathbb{M}_{\mathbb{V}} F_S(.))] + ({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.)) \frac{\partial[({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.)) \cdot (\mathbb{M}_{\mathbb{V}} F_S(.))]}{\partial u}\right] \tag{40}$$

$$= k_{C_M} \mathbb{M}_{\mathbb{V}}^{\top}\left[-\mathbb{M}_{\mathbb{P}} \frac{\partial F_S(.)}{\partial u}[({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.)) \cdot (\mathbb{M}_{\mathbb{V}} F_S(.))] + ({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.))\left((\mathbb{M}_{\mathbb{V}} F_S(.))^{\top} \frac{\partial({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.))}{\partial u} + ({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.))^{\top} \frac{\partial(\mathbb{M}_{\mathbb{V}} F_S(.))}{\partial u}\right)\right] \tag{41}$$

$$= k_{C_M} \mathbb{M}_{\mathbb{V}}^{\top}\left[-\mathbb{M}_{\mathbb{P}} \frac{\partial F_S(.)}{\partial u}[({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.)) \cdot (\mathbb{M}_{\mathbb{V}} F_S(.))] + ({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.))\left(-(\mathbb{M}_{\mathbb{V}} F_S(.))^{\top} \mathbb{M}_{\mathbb{P}} \frac{\partial F_S(.)}{\partial u} + ({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.))^{\top} \mathbb{M}_{\mathbb{V}} \frac{\partial F_S(.)}{\partial u}\right)\right] \tag{42}$$

$$= k_{C_M} \mathbb{M}_{\mathbb{V}}^{\top}\left[-\mathbb{M}_{\mathbb{P}} \frac{\partial F_S(.)}{\partial u}[({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.)) \cdot (\mathbb{M}_{\mathbb{V}} F_S(.))] + ({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.))\left(-(\mathbb{M}_{\mathbb{V}} F_S(.))^{\top} \mathbb{M}_{\mathbb{P}} + ({}^2x^- - \mathbb{M}_{\mathbb{P}} F_S(.))^{\top} \mathbb{M}_{\mathbb{V}}\right) \frac{\partial F_S(.)}{\partial u}\right] \tag{43}$$

2.2.5 Computing $\frac{\partial F_S(t_{c_2}(u), t_{c_1}(u), C_S^D(.))}{\partial u}$:

We take the partial derivative of Equation 6:

$$\frac{\partial F_S(t_{c_2}(u), t_{c_1}(u), C_S^D(.))}{\partial u} = \frac{\partial\left[C_S^D(.) + (t_{c_2} - t_{c_1})\mathbb{M}C_S^D(.) + \begin{bmatrix} -\frac{1}{2} g\mu(t_{c_2} - t_{c_1})^2 \\ -\frac{1}{2} g\mu(t_{c_2} - t_{c_1})^2 \\ 0 \\ -g\mu(t_{c_2} - t_{c_1}) \\ -g\mu(t_{c_2} - t_{c_1}) \\ 0 \end{bmatrix}\right]}{\partial u} \tag{44}$$

$$= \frac{\partial C_S^D(.)}{\partial u} + \frac{\partial\left[\mathbb{M}C_S^D(.)(t_{c_2} - t_{c_1})\right]}{\partial u} - \frac{\partial \begin{bmatrix} \frac{1}{2} g\mu(t_{c_2} - t_{c_1})^2 \\ \frac{1}{2} g\mu(t_{c_2} - t_{c_1})^2 \\ 0 \\ g\mu(t_{c_2} - t_{c_1}) \\ g\mu(t_{c_2} - t_{c_1}) \\ 0 \end{bmatrix}}{\partial u} \tag{45}$$

$$= \frac{\partial C_S^D(.)}{\partial u} + \frac{\partial\left[\mathbb{M}C_S^D(.)(t_{c_2} - t_{c_1})\right]}{\partial u} - \begin{bmatrix} g\mu(t_{c_2} - t_{c_1})\partial(t_{c_2} - t_{c_1})/\partial u \\ g\mu(t_{c_2} - t_{c_1})\partial(t_{c_2} - t_{c_1})/\partial u \\ 0_{1\times 6} \\ g\mu\partial(t_{c_2} - t_{c_1})/\partial u \\ g\mu\partial(t_{c_2} - t_{c_1})/\partial u \\ 0_{1\times 6} \end{bmatrix} \tag{46}$$

$$= \frac{\partial C_S^D(.)}{\partial u} + \left[\frac{\partial(\mathbb{M}C_S^D(.))}{\partial u}(t_{c_2} - t_{c_1}) + (\mathbb{M}C_S^D(.))\frac{\partial(t_{c_2} - t_{c_1})}{\partial u}\right] - g\mu \begin{bmatrix} (t_{c_2} - t_{c_1}) \\ (t_{c_2} - t_{c_1}) \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \frac{\partial(t_{c_2} - t_{c_1})}{\partial u} \tag{47}$$

$$= \frac{\partial C_S^D(.)}{\partial u} + (t_{c_2} - t_{c_1})\mathbb{M}\frac{\partial C_S^D(.)}{\partial u} + (\mathbb{M}C_S^D(.))\left(\frac{\partial t_{c_2}}{\partial u} - \frac{\partial t_{c_1}}{\partial u}\right) - g\mu\begin{bmatrix}(t_{c_2}-t_{c_1})\\(t_{c_2}-t_{c_1})\\0\\1\\1\\0\end{bmatrix}\left(\frac{\partial t_{c_2}}{\partial u} - \frac{\partial t_{c_1}}{\partial u}\right) \quad (48)$$

$$= \left(I_{6\times6} + (t_{c_2} - t_{c_1})\mathbb{M}\right)\frac{\partial C_S^D(.)}{\partial u} + \left(\mathbb{M}C_S^D(.) - g\mu\begin{bmatrix}(t_{c_2}-t_{c_1})\\(t_{c_2}-t_{c_1})\\0\\1\\1\\0\end{bmatrix}\right)\left(\frac{\partial t_{c_2}}{\partial u} - \frac{\partial t_{c_1}}{\partial u}\right) \quad (49)$$

$$(50)$$

2.2.6 Computing $\frac{\partial C_S^D(F_F(.))}{\partial u}$:

This one is simple - we take derivative of Equation 13:

$$\frac{\partial C_S^D(F_F(.))}{\partial u} = \frac{\partial[\mathbb{M}_C F_F(.)]}{\partial u} \quad (51)$$

$$= \mathbb{M}_C\frac{\partial F_F(.)}{\partial u} \quad (52)$$

2.2.7 Computing $\frac{\partial F_F(t_{c_1}(u),0,{}^1q_0(u))}{\partial u}$:

This derivative is covered in the free-flight document and copied from there:

$$\frac{\partial F(t_{c_1}(u),t_0,{}^1q_0(u))}{\partial u} = \frac{\partial\left[{}^1q_0(u)+\mathbb{M}^1q_0(u)t_{c_1}(u)+\begin{bmatrix}0\\0\\-\frac{1}{2}gt_{c_1}(u)^2\\0\\0\\-gt_{c_1}(u)\end{bmatrix}\right]}{\partial u} \quad (53)$$

$$= \frac{\partial^1 q_0(u)}{\partial u} + \frac{\partial[\mathbb{M}^1q_0(u)t_{c_1}(u)]}{\partial u} + \frac{\partial\begin{bmatrix}0\\0\\-\frac{1}{2}gt_{c_1}(u)^2\\0\\0\\-gt_{c_1}(u)\end{bmatrix}}{\partial u} \quad (54)$$

$$= \left[I_{6\times6}, 0_{6\times1}\right] + \left[\frac{\partial M^1q_0(u)}{\partial u}t_{c_1}(u) + \mathbb{M}^1q_0(u)\frac{\partial(t_{c_1}(u))}{\partial u}\right] + \begin{bmatrix}0_{1\times7}\\0_{1\times7}\\\partial[-\frac{1}{2}gt_{c_1}(u)^2]/\partial u\\0_{1\times7}\\0_{1\times7}\\\partial[-gt_{c_1}(u)]/\partial u\end{bmatrix} \quad (55)$$

$$= \left[I_{6\times6}, 0_{6\times1}\right] + \mathbb{M}\left[I_{6\times6}, 0_{6\times1}\right]t_{c_1}(u) + \mathbb{M}^1q_0(u)\frac{\partial t_{c_1}(u)}{\partial u} + \begin{bmatrix}0_{1\times7}\\0_{1\times7}\\-gt_c(u)^{\partial t_{c_1}(u)/\partial u}\\0_{1\times7}\\0_{1\times7}\\-g^{\partial t_{c_1}(u)/\partial u}\end{bmatrix} \quad (56)$$

2.3 Analysis

The dependency between the derivative and state computations is seen as follows. An important question is whether these computations are as separable as possible semanticly. That is, have we parameterized this motion such that we can generate atomic functions that can be used to generate new motions?

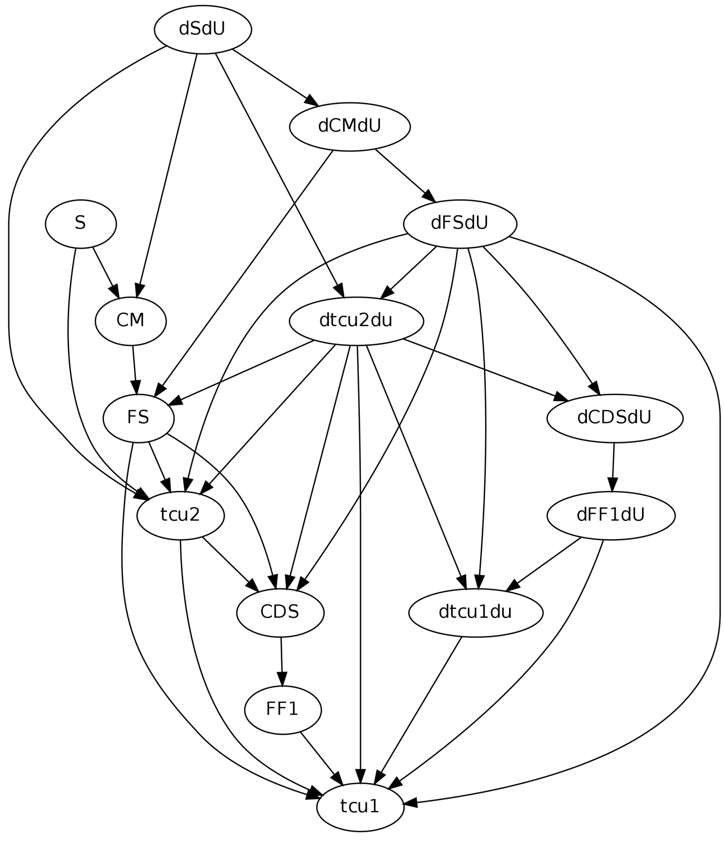

Figure 1: Dependency between the variables

We seek the answer to this now:

1. $\frac{\partial S}{\partial u}$: This only depends on $\frac{\partial C_M}{\partial u}$, $\frac{\partial t_{c_2}(u)}{\partial u}$ and C_M which makes sense since C_M is the initial state of the flight and the $(t_f - t_{c_2}(u))$ is the flight duration. *However, it might be possible to just generate a new variable that express the flight motion and remove the dependency on* $t_{c_2}(u)$.

2. $\frac{\partial C_M}{\partial u}$: This only depends on F_S and its partial derivative which makes sense.

3. $\frac{\partial F_S}{\partial u}$: Similar problem as the first one. The duration of the sliding depends on $(t_{c_2}(u) - t_{c_1}(u))$ which means additional parameters need to be fed into the computation of the derivative instead of just having a t_s, slide time variable. Ideally, this would only depend on C_S^D, $\frac{\partial C_S^D}{\partial u}$, t_s, and $\frac{\partial t_s}{\partial u}$.

4. $\frac{\partial t_{c_2}(u)}{\partial u}$: Again the time slide t_s.

5. $\frac{\partial C_S^D}{\partial u}$: Straightforward.

6. $\frac{\partial F_{F_1}}{\partial u}$: Depends only on $t_{c_1}(u)$ and its partial, as it should.

Please see slidingBookClean.pdf for the necessary changes.