Closed-Form Rigid Body Orientations

Can Erdogan

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1 Introduction

This document attempts to explain the closed-form rigid body orientations in a brief and simpler manner and it is based on Zon and Schofield's work [1]. We start by introducing notation and basics for rigid body dynamics. The cited work assumes there is no external forces or torques on the rigid body throughout the motion and analyzes the motion based off the initial conserved linear and angular momentums.

1.1 Rigid bodies

A rigid body is assumed to be a collection of N smaller particles with coordinates $\bar{\boldsymbol{r}}_i$ in the body frame and mass m_i. The origin is at the center of mass such that

$$\sum_i m_i \bar{\boldsymbol{r}}_i = 0 \tag{1}$$

Furthermore, the body frame is assumed to be aligned such that the inertia tensor $\tilde{\mathbf{I}}$ has the form:

$$\tilde{\mathbf{I}} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{2}$$

where the constants I_1, I_2 and I_3 are called the *principal moments of inertia*.

Let $\boldsymbol{R}$ denote the position of the center of mass in the world frame and let $\mathbf{A}$ be the rotation matrix that transforms vectors in the world frame to the body frame. Now, we define the location of a particle in the world frame:

$$\boldsymbol{r}_i = \boldsymbol{R} + \mathbf{A}^\mathsf{T} \hat{\boldsymbol{r}}_i \tag{3}$$

where the inverse of the rotation $\mathbf{A}^{-1} = \mathbf{A}^\mathsf{T}$ since the *attitude matrix* $\mathbf{A}$ is orthonormal and in $SO(3)$.

Next, we define the angular velocity $\boldsymbol{\omega} = \{\omega_1, \omega_2, \omega_3\}$ such that its direction is the axis around which the body rotates and its magnitude is the rate of rotation. For each particle, we define the linear velocity:

$$\boldsymbol{v}_i = \boldsymbol{V} + \boldsymbol{\omega} \times (\boldsymbol{r}_i - R) \tag{4}$$

$$= \boldsymbol{V} + \mathbf{W}(\boldsymbol{\omega}) A^\mathsf{T} \tilde{\boldsymbol{r}}_i \tag{5}$$

where $\boldsymbol{V} = \dot{\boldsymbol{R}}$ is the linear velocity of the body, and we can replace the cross product with a skew symmetric matrix for infinitesmall rotation, $\mathbf{W}$. By taking the derivative of Equation 3 and combining it with Equation 5, we can obtain the following useful observation:

$$\dot{\mathbf{A}}^\mathsf{T} \mathbf{A} = \mathbf{W}(\boldsymbol{\omega}) \tag{6}$$

Finally, we introduce the angle-axis rotation with the symbol $\mathbf{U}(\psi \hat{\boldsymbol{n}})$ where $\hat{\boldsymbol{n}}$ is a normalized vector and ψ is the angle for the amount of rotation.

1.2 Dynamics

Let $\boldsymbol{F}$ be the sum of forces on the body, $\boldsymbol{P} = M\boldsymbol{V}$ (with $M = \sum_i m_i$) is the linear momentum, $\boldsymbol{\tau}$ is the total torque with respect to the center of mass, and $\boldsymbol{L} = \mathbf{I}\boldsymbol{\omega}$ is the angular momentum. These terms relate to each other as follows:

$$\boldsymbol{F} = \dot{\boldsymbol{P}} = M\dot{\boldsymbol{V}} \tag{7}$$

$$\boldsymbol{\tau} = \dot{\boldsymbol{L}} = \frac{d}{dt}(\mathbf{I}\boldsymbol{\omega}) \tag{8}$$

We can get an insight into how the torque affects both the body orientation and the angular velocity by continuing the differentiation above in the body frame:

$$\tilde{\boldsymbol{\tau}} = \mathbf{A}\dot{\mathbf{A}}^{\mathsf{T}}\tilde{\mathbf{I}}\tilde{\boldsymbol{\omega}} + \tilde{\mathbf{I}}\dot{\tilde{\boldsymbol{\omega}}} \tag{9}$$

$$= \mathbf{W}(\tilde{\boldsymbol{\omega}})\tilde{\mathbf{I}}\tilde{\boldsymbol{\omega}} + \tilde{\mathbf{I}}\dot{\tilde{\boldsymbol{\omega}}} \tag{10}$$

Using Equations 6 and 10, we can derive a relationship between the attitude matrix $\mathbf{A}$, and the angular velocity in the local frame $\tilde{\boldsymbol{\omega}} = A\boldsymbol{\omega}$:

$$\dot{\mathbf{A}} = -\mathbf{W}(\tilde{\boldsymbol{\omega}})\mathbf{A} \tag{11}$$

2 Solving for the attitude matrix A in the absence of torques

Given that there are no external torques throughout the motion, we can rewrite Equation 10 as:

$$\tilde{\mathbf{I}}\dot{\tilde{\boldsymbol{\omega}}} = -\tilde{\boldsymbol{\omega}} \times \tilde{\mathbf{I}}\tilde{\boldsymbol{\omega}} \tag{12}$$

We can also expand this equation with the components ω_i as they will be used later on in the solution:

$$\begin{aligned} I_1\dot{\tilde{\omega}}_1 &= \tilde{\omega}_2\tilde{\omega}_3(I_2 - I_3) \\ I_2\dot{\tilde{\omega}}_2 &= \tilde{\omega}_3\tilde{\omega}_1(I_3 - I_1) \\ I_3\dot{\tilde{\omega}}_3 &= \tilde{\omega}_1\tilde{\omega}_2(I_1 - I_2) \end{aligned} \tag{13}$$

Now, the goal is to solve for $\mathbf{A}$ in the differential equation in Equation 11 so that the following holds:

$$\mathbf{A}(t) = \mathbf{P}(t)\mathbf{A}(0) \tag{14}$$

where we assume at time $t = 0$, the body and the world frame coincides. Note that placing the world frame to the initial body state implies that $\mathbf{P}(t)$ relates the body frame to the world frame at any time t: $\tilde{\boldsymbol{v}} = \mathbf{P}(t)\boldsymbol{v}$ for any vector v. When the body frame is expressed in the world frame with $\mathbf{P}$, the initial angular momentum preserved over time can be any vector in $\mathbb{R}^3$:

$$\boldsymbol{L}(t) = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \tilde{\mathbb{L}}(0) \tag{15}$$

However, we can make the solution expressions even simpler by changing the choice of frame orientation such that the angular momentum $\boldsymbol{L} = [0, 0, L]^{\mathsf{T}}$ for some L. The matrix ${\mathbf{T}'_1}^{\mathsf{T}}(0)$ does the trick:

$${\mathbf{T}'_1}^{\mathsf{T}}(0) = \underbrace{\begin{pmatrix} \frac{\tilde{L}_3(0)}{L} & 0 & -\frac{\tilde{L}_\perp(0)}{L} \\ 0 & 1 & 0 \\ \frac{\tilde{L}_\perp(0)}{L} & 0 & \frac{\tilde{L}_3(0)}{L} \end{pmatrix}}_{\text{rotation to get } \boldsymbol{L}'_x(0) = 0} \underbrace{\begin{pmatrix} \frac{\tilde{L}_1(0)}{\tilde{L}_\perp(0)} & \frac{\tilde{L}_2(0)}{\tilde{L}_\perp(0)} & 0 \\ -\frac{\tilde{L}_2(0)}{\tilde{L}_\perp(0)} & \frac{\tilde{L}_1(0)}{\tilde{L}_\perp(0)} & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\text{rotation to get } \boldsymbol{L}'_y(0) = 0} = \begin{pmatrix} \frac{\tilde{L}_1(0)\tilde{L}_3(0)}{\tilde{L}_\perp(0)L} & \frac{\tilde{L}_2(0)\tilde{L}_3(0)}{L\tilde{L}_\perp(0)} & -\frac{\tilde{L}_\perp(0)}{L} \\ -\frac{\tilde{L}_2(0)}{\tilde{L}_\perp(0)} & \frac{\tilde{L}_1(0)}{\tilde{L}_\perp(0)} & 0 \\ \frac{\tilde{L}_1(0)}{L} & \frac{\tilde{L}_2(0)}{L} & \frac{\tilde{L}_3(0)}{L} \end{pmatrix} \tag{16}$$

where $\tilde{L}_\perp = \sqrt{((\tilde{L}_1)^2 + (\tilde{L}_2)^2)}$.

We can now define the solution to the differential in Equation 11 in the simpler invariant frame:

$$\mathbf{P'} = P{\mathbf{T}'_1}^{\mathsf{T}}(0) \tag{17}$$

where the initial condition $\mathbf{P'}(0) = {\mathbf{T}'_1}^{\mathsf{T}}(0)$ is necessary. Let $\hat{\boldsymbol{u}}_1$, $\hat{\boldsymbol{u}}_2$ and $\hat{\boldsymbol{u}}_3$ be the columns of $\mathbf{P'}$. In this new frame, by design, we have:

$$\hat{\boldsymbol{u}}_3 = \mathbf{P'}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{\mathbf{P'}\boldsymbol{L'}}{L} = \frac{\tilde{\boldsymbol{L}}}{L} \tag{18}$$

where the numerator and the denominator are known input values. We can also express the first two columns as sinusoidal functions of a single time-dependent angle ψ. Note that these two columns lie in a plane perpendicular to $\hat{\boldsymbol{u}}_3$. We define the following two orthogonal unit vectors to span that plane:

$$\hat{\boldsymbol{e}}_1 = \begin{pmatrix} -\frac{\tilde{L}_1\tilde{L}_3}{L\tilde{L}_bot} \\ \frac{\tilde{L}_2\tilde{L}_3}{L\tilde{L}_bot} \\ -\frac{\tilde{L}_\perp}{L} \end{pmatrix} \qquad \hat{\boldsymbol{e}}_2 = \begin{pmatrix} -\frac{\tilde{L}_2}{\tilde{L}_bot} \\ \frac{\tilde{L}_1}{\tilde{L}_bot} \\ 0 \end{pmatrix} \tag{19}$$

where $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$ actually constitute the first two columns of the rotation matrix $\mathbf{T}'_1(0)$. Now, we can define the first two columns of $\mathbf{P'}$, $\hat{\boldsymbol{u}}_1$ and $\hat{\boldsymbol{u}}_2$ in terms of ψ:

$$\hat{\boldsymbol{u}}_1 = \hat{\boldsymbol{e}}_1 cos(\psi) - \hat{\boldsymbol{e}}_2 sin(\psi) \tag{20}$$

$$\hat{\boldsymbol{u}}_2 = \hat{\boldsymbol{e}}_1 sin(\psi) + \hat{\boldsymbol{e}}_2 cos(\psi) \tag{21}$$

and the choice of orthogonal unit vectors allows us to ensure $\psi(0) = 0$. In a closer inspection, we can see that we have parametrized $\mathbf{P'}$ with the angle ψ which will represent the rotation around the rotated z axis with the matrix $\mathbf{T}'_1$. Subsequently, we can write the two solutions for $\mathbf{P'}$ in the invariant frame and $\mathbf{P}$ in the world frame:

$$\mathbf{P'} = \mathbf{T}'_1\mathbf{U}(-\psi\hat{\boldsymbol{z}}) \tag{22}$$

$$\mathbf{P} = \mathbf{T}'_1\mathbf{U}(-\psi\hat{\boldsymbol{z}}){\mathbf{T}'_1}^{\mathsf{T}}(0) \tag{23}$$

2.1 Computation of the matrix $\mathbf{P'}$

Although we skip the majority of the math in [1], we still need some background in complex analysis.

2.1.1 Jacobi Theta Function

We start by defining the *incomplete elliptical integral of the first kind*, $F(x|m)$ where m is the elliptic parameter that controls the how similar to sine and cosine curves the periodic elliptic function is:

$$\mathbf{F}(x|m) = \int_0^x \frac{dt}{\sqrt{1-mt^2}\sqrt{1-t^2}} \tag{24}$$

Note that the elliptic parameter m is a constant computed from the initial body state as follows:

$$m = \frac{(L^2 - 2I_3E_R)(I_1 - I_2)}{(L^2 - 2I_1E_R)(I_3 - I_2)} \tag{25}$$

Along with m, we can define three constants, namely the *quarter period* $K = F(1|m)$, the *complementary quarter period* $K' = F(1|1-m)$ and the *nome* $q = e^{-\pi\frac{K}{K'}}$. These constants now help us define the *Jacobi theta function*, one of the three main components of the ψ computation:

$$\mathbf{\Theta}(u|m) = 2q^{1/4}(m)\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}(m) sin[(2n+1)u] \tag{26}$$

2.1.2 Jacobi Elliptic Functions

Before we discuss the input to the theta function, we need to provide some context. First, we assume that the principal moments of inertia are either $I_1 > I_2 > I_3$ if $E_R > \frac{L^2}{2I_2}$ or $I_1 < I_2 < I_3$ otherwise. To ensure this is the case, the initial orientation frame can be offseted.

Let cn, sn and dn be the *Jacobi elliptic functions.* Using them, we can show the results for the angular velocities for the differential constraint:

$$\tilde{\omega}_1 = \tilde{\omega}_{1m}\mathrm{cn}(\omega_p t + \epsilon|m) \tag{27}$$

$$\tilde{\omega}_2 = \tilde{\omega}_{2m}\mathrm{sn}(\omega_p t + \epsilon|m) \tag{28}$$

$$\tilde{\omega}_3 = \tilde{\omega}_{3m}\mathrm{dn}(\omega_p t + \epsilon|m) \tag{29}$$

where the following constants are necessary:

$$\tilde{\omega}_{1m} = sgn(\tilde{\omega}_1(0))\sqrt{\tfrac{L^2-2I_3E_R}{I_1(I_1-I_3)}} \tag{30}$$

$$\tilde{\omega}_{2m} = -sgn(\tilde{\omega}_1(0))\sqrt{\tfrac{L^2-2I_3E_R}{I_2(I_2-I_3)}} \tag{31}$$

$$\tilde{\omega}_{3m} = sgn(\tilde{\omega}_3(0))\sqrt{\tfrac{L^2-2I_1E_R}{I_3(I_3-I_1)}} \tag{32}$$

$$\omega_p = sgn(I_2 - I_3)sgn(\tilde{\omega}_3(0))\sqrt{\tfrac{(L^2-2I_1E_R)(I_3-I_2)}{I_1I_2I_3}} \tag{33}$$

$$\epsilon = F(\tfrac{\tilde{\omega}_2(0)}{\tilde{\omega}_{2m}}|m) \tag{34}$$

Finally, we also need the following constant η:

$$\eta = sgn(\tilde{\omega}_{3m})K' - F(\tfrac{I_3\tilde{\omega}_{3m}}{L}|1-m) \tag{35}$$

2.1.3 Computing $sin(\psi)$ and $cos(\psi)$:

As shown in Equations 20-21, to compute the first two columns of $\mathbf{P'}$, we only need $sin(\psi)$ and $cos(\psi)$, not ψ. Here, we present the solution for these two expressions and next, we expand on the subterms:

$$cos(\psi) = \frac{cos(A_1+A_2t)Re(\mathbf{\Theta}(u|m))+sin(A_1+A_2t)Im(\mathbf{\Theta}(u|m))}{\sqrt{Re(\mathbf{\Theta}(u|m))^2+Im(\mathbf{\Theta}(u|m))^2}} \tag{36}$$

$$sin(\psi) = \frac{sin(A_1+A_2t)Re(\mathbf{\Theta}(u|m))-cos(A_1+A_2t)Im(\mathbf{\Theta}(u|m))}{\sqrt{Re(\mathbf{\Theta}(u|m))^2+Im(\mathbf{\Theta}(u|m))^2}} \tag{37}$$

where $Re(.)$ and $Im(.)$ returns the real and imaginary parts of a complex number. Finally, the input to the theta function is $u = \frac{\pi}{2K}(\omega_p t + \epsilon - i\eta)$ and with this, we can evaluate these two parts:

$$Re(\mathbf{\Theta}(u|m)) = 2q^{1/4}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}cosh\big(\tfrac{(2n+1)\pi\eta}{2K}\big)sin\big(\tfrac{(2n+1)\pi(\omega_p t+\epsilon)}{2K}\big) \tag{38}$$

$$Im(\mathbf{\Theta}(u|m)) = -2q^{1/4}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}sinh\big(\tfrac{(2n+1)\pi\eta}{2K}\big)cos\big(\tfrac{(2n+1)\pi(\omega_p t+\epsilon)}{2K}\big) \tag{39}$$

Next, we define the constant A_1 :

$$A_1 = \arg(\mathbf{\Theta}(\tfrac{\pi}{2K}(\epsilon - i\eta)|m)) \tag{40}$$

$$= n\pi + arctan\Big(\frac{Im(\mathbf{\Theta}(\frac{\pi}{2K}(\epsilon-i\eta)|m))}{Re(\mathbf{\Theta}(\frac{\pi}{2K}(\epsilon-i\eta)|m))}\Big) \tag{41}$$

where the real and imaginary parts of the theta function with input $u_2 = \frac{\pi}{2K}(\epsilon - i\eta)$ can be computed very similar to Equations 38 and 39, by removing the term $w_p t$. The value n is defined as follows:

$$n = \begin{cases} 0, & \text{if } Re(\mathbf{\Theta}(u_2|m)) > 0 \\ 1, & \text{if } Re(\mathbf{\Theta}(u_2|m)) < 0 \quad \text{and} \quad Im(\mathbf{\Theta}(u_2|m)) > 0 \\ -1, & \text{otherwise.} \end{cases}$$

Finally, we can define the last constant term A_2 where $\xi = e^{\pi\eta/K}$:

$$A_2 = \frac{L}{I_1} + \frac{\pi\omega_p}{2K}\left[\frac{\xi+1}{\xi-1} - 2\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}(\xi^n - \xi^{-n})\right] \tag{42}$$

2.2 Computation of the quaternion

In the previous subsection, we focused on computing the $cos(\psi)$ and $sin(\psi)$ so that the first two columns of the matrix $U(-\psi\hat{\boldsymbol{z}})$ can be formed. If we were to use the quaternion representation, then, we would be more interested in computing ψ directly:

$$\psi = A_1 + A_2 t - \arg \mathbf{\Theta}\big(\tfrac{\pi}{2K}(\omega_p t + \epsilon - i\eta)|m\big) \tag{43}$$

where the Jacobi theta function can be expanded with a summation as shown in Equation 26.

3 Angular velocities at time t

To compute the rotation for the invariant frame, we had to calculate the angular velocities $\tilde{\omega}_1$, $\tilde{\omega}_2$ and $\tilde{\omega}_3$ in the body frame in Equations 27-29. To obtain these values in the world frame, we need to change the reference frame by multiplying with $\mathbf{P}'$:

$$\boldsymbol{\omega} = \mathbf{P}' \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \end{bmatrix} \tag{44}$$

$$= \mathbf{T}_1' U(-\psi\hat{\boldsymbol{z}}) \begin{bmatrix} sgn(\tilde{\omega}_1(0))\sqrt{\frac{L^2-2I_3E_R}{I_1(I_1-I_3)}}\, cn(w_p t + \epsilon|m) \\ -sgn(\tilde{\omega}_1(0))\sqrt{\frac{L^2-2I_3E_R}{I_2(I_2-I_3)}}\, sn(w_p t + \epsilon|m) \\ sgn(\tilde{\omega}_3(0))\sqrt{\frac{L^2-2I_1E_R}{I_3(I_3-I_1)}}\, dn(w_p t + \epsilon|m) \end{bmatrix} \tag{45}$$

4 Conclusion

The goal was to compute $\mathbf{P}(t)$ such that we can compute the attitude (rotation) matrix $\mathbf{A}(t)$ at any time t. We observed that at a rotated frame, the *invariant frame* where the angular momentum is through a single axis, the new difference matrix $\mathbf{P}(t)'$ is simpler to compute. The "single axis" assumption allowed us to fix the last column $\hat{\boldsymbol{u}}_3$ of the matrix and parameterize the first two with angle ψ. Using some complex analysis, we found a closed form solution for $cos(\psi)$ and $sin(\psi)$ (Equations 36 and 37) which are in turn functions of the theta function, time t, and constants A_1 and A_2 as shown in Equations 38-41.

Figure 1 below demonstrates the accuracy of the model in comparison to the simulation, whether regardless of the gravity acting on the body. The model and gravityless simulation starts diverging after more than 1.5 seconds with the chosen inertia and initial angular velocities. Note that the two cases with and without the gravity follow each other perfectly under the collision with ground where a sudden change in angle can be observed.

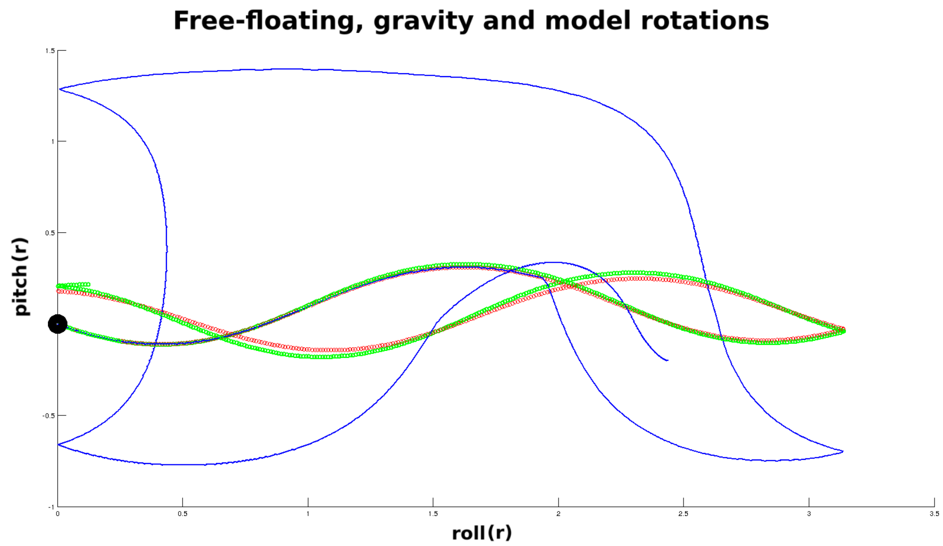

Figure 1: Comparison of rotation motions between (1) simulation without gravity (red-circle), (2) simulation with gravity (blue-line), (3) model (green circle). Simulation starts at the black circle.

References

[1] Van Zon, Ramses and Schofield, Jeremy, *Numerical implementation of the exact dynamics of free rigid bodies.* Journal of Computational Physics, 225, 1, 145–164, 2007, Elsevier.